JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **27**, No. 1, 2014
http://dx.doi.org/10.14403/jcms.2014.27.1.35

A NOTE ON A QUESTION RELATED TO KOLMOGOROV FORWARD EQUATION

Joongul Lee*

Abstract. We examine a question raised in [1] regarding existence and uniqueness of the steady-state solution of certain type of Kolmogorov forward equation in this paper, and prove that the question holds for S^1 by giving an explicit solution.

1. Introduction

We first review the question raised in [1].

Let $\mathcal{M}$ be an orientable Riemannian manifold of dimension n, whose inner product at each point is denoted by $\langle\ ,\ \rangle$. The inner product gives us a canonical isomorphism between the 1-forms on $\mathcal{M}$ and the vector fields on $\mathcal{M}$. For convenience we denote the isomorphism and its inverse by a *tilde*. Then if f is a function on $\mathcal{M}$, its gradient is the vector field

$$\nabla f = \widetilde{df}.$$

And if E is a vector field on $\mathcal{M}$, its divergence is the function

$$\nabla\cdot E = * d * \widetilde{E}$$

where $*$ is the Hodge star operator. (cf. [2])

Question 1.1. Assume $\mathcal{M}$ is compact. For a vector field X on $\mathcal{M}$, is it possible to fi is it possible to find unique vector fields E, F with the

1. $X = E + F$,
2. $F = -\nabla\phi$ for some function ϕ on $\mathcal{M}$,
3. $\langle E, F\rangle + \nabla\cdot E = 0$ at every point of $\mathcal{M}$.

As explained in [1], one can derive the steady-state solution of certain type of Kolmogorov forward equation from such decomposition.

Received August 21, 2013; Accepted January 16, 2014.
2010 Mathematics Subject Classification: Primary 58J65; Secondary 53A45.
Key words and phrases: Kolmogorov forward equation, Fokker-Planck equation.
This work was supported by 2012 Hongik University Research Fund.

2. The main result

THEOREM 2.1. *For $\mathcal{M} = S^1$, the question has an affirmative answer for every vector field.*

Proof. Let $p : \mathbb{R} \to S^1$ be the covering map $p(x) = (\cos x, \sin x)$. p is a local isometry and is periodic with period 2π. Therefore we may identify the vector fields on S^1 with periodic functions (with period 2π) on $\mathbb{R}$, and under such identification the gradient and divergence are just the derivative. A periodic function g on $\mathbb{R}$ is a derivative of a periodic function if and only if $\int_0^{2\pi} g\,dx = 0$.

Let $f(x)$ be the given periodic function, and consider the differential equation

$$\frac{dy}{dx} + fy - y^2 = 0. \tag{2.1}$$

This is a Bernoulli equation and it can be transformed into a first order linear differential equation. Let $\alpha = \int_0^{2\pi} f(t)\,dt$. Choose a primitive $F(x)$ of $f(x)$ (i.e. $\frac{dF}{dx} = f(x)$), and let $I(x) = e^{-F(x)}$. Multiplying I on the differential equation and integrating, we get the general solution $y = 0$ or

$$y(x) = \frac{I(x)}{c - \int_0^x I(t)\,dt} \tag{2.2}$$

where c is a constant.

To find a periodic solution, first assume $\alpha \neq 0$. Setting $y(0) = y(2\pi)$ on Equation (2.2) gives

$$c = \frac{\int_0^{2\pi} I(t)\,dt}{1 - e^{-\alpha}},$$

and so

$$y(x) = \frac{(1 - e^{-\alpha})I(x)}{\int_0^{2\pi} I(t)\,dt - (1 - e^{-\alpha}) \int_0^x I(t)\,dt}. \tag{2.3}$$

Equation (2.3) makes sense and gives the periodic solution $y = 0$ when $\alpha = 0$. Also note that the denominator of Equation (2.3) is never 0.

To prove that Equation (2.3) gives a periodic solution, let $z(x) = y(x + 2\pi)$. $z(x)$ is also a solution of the given differential equation (2.1) because f is periodic. As $z(0) = y(2\pi) = y(0)$, it follows from the uniqueness of the solution of differential equation that $z = y$.

$f - y$ is a derivative of a periodic function because either f itself is a derivative (when $\alpha = 0$) or

$$f - y = -\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\{-\log|y|\},$$

together with the fact that $y(x) \neq 0$ for $x \in \mathbb{R}$ when $\alpha \neq 0$. Finally we note that $y(x)$ in Equation (2.3) is independent of the choice of F. $\square$

References

[1] Joongul Lee, *On a question related to kolmogorov forward equation*, Honam Math. J. **35** (2013), no. 4, 679-681.
[2] Frank W. Warner, *Foundations of differentiable manifolds and Lie groups*, Scott, Foresman and Co., Glenview, Ill.-London, 1971.

*
Department of Mathematics Education
Hongik University
Seoul 121-791, Republic of Korea
E-mail: jglee@hongik.ac.kr